# Electromagnetic properties of the linear oscillating currents flowing through toroidal knots

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Magnetic and toroid formfactors as well as the radiation intensity for a linear oscillating current flowing through a (knotted) toroidal spiral with a fractional number $n=q/p(p,q=\text{integers})$ of turns of winding are calculated. Some topological properties of such current lines are analyzed by studying the isotopic deformations when the ratio $\varepsilon$ of the torus radii varies from 0 to 1 and a critical value is found for it, $\varepsilon_c=p^2/(p^2+q^2)$, at which the Calugareanu invariant $K$ (integer number) jumps by $q$ units.

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We report here some results of an investigation that happens to involve both some aspects of knot theory, especially those related to Calugareanu invariants [1], and the intensively explored domain of toroid moments. Originally discovered theoretically by Zeldovich [2] (under the name of ''anapole''), the toroid dipole (a specific electromagnetic moment of a dipole type, different, however, from the usual electric and magnetic dipoles) led through clarification and generalization [3] to an entire class of toroid multipoles needed in order to realize a correct and complete multipole characterization of the most general type of source in both classical and quantum electrodynamics. Over the years, the activity in the field of toroid moments increased [see Ref. [4] and references therein], the first experimental evidence of a nuclear spin-dependent contribution to atomic parity nonconservation, on account of a nuclear intrinsic toroid dipole [5], stimulating further the investigations. Intuitively, the toroid dipole is a toroidally-wound solenoid shrinked to a point. It singles out a direction in space exactly as do the usual electric and magnetic dipoles.

The study of toroid moments revealed very interesting features, previously unnoted, in many areas of physics, and the analysis of toroidal structures even in classical electrodynamics appears quite important in this context. In Ref. [6] toroidal currents flowing over the surface of a torus were considered and interesting results were found. If surface currents provide such a good object to look for particularities of toroidal structures, why not look into linear currents flowing through a toroidal spiral with a certain number $n$ of turns of winding. While for an integer $n$, the spiral can be countinuously deformed to a circle, for fractional $n$ the spiral is a toroidal knot. Next we shall say something interesting about the knots and about an oscillating electric current going through such a knotted wire.

The parametric equations of the toroidal spiral considered are

$$x(v)=[R_T+r_T\cos(nv)]\cos(v),\quad n=\frac{q}{p}, \tag{1}$$

$$y(v)=[R_T+r_T\cos(nv)]\sin(v),\quad v\in[0,2p\pi),$$

$$z(v)=r_T\sin(nv),\quad \varepsilon\equiv\frac{r_T}{R_T},\quad \varepsilon\in[0,1].$$

$R_T$ and $r_T$ are the large and small radii of the torus, while $p$ and $q$ are integer numbers, so that $n$ may be fractional. When $n$ is integer, the spiral is a ''normal'' one, i.e., unknotted. Otherwise (for $q$, $p$ mutually prime, $q>1$) it is always knotted.

We have studied, both analytically and numerically, deformations of the spiral given by Eqs. (1) by varying $\varepsilon$ in the whole admissible region $\varepsilon\in[0,1]$, i.e., going from very thin tori to those of maximum possible thickness $r_T=R_T$. For such deformations of a (knotted) closed curve $C$ given by equations such as Eqs. (1), parametrically ($\varepsilon$) dependent, $x=x(t),y=y(t),z=z(t)$, $t\in[0,a]$, we have evaluated the remarkable Calugareanu invariant of the third order (i.e., $x(t),y(t),z(t)$ have continuous derivatives upto the third order)

$$K=J+Y,$$

$$J=\frac{1}{4\pi}\int_C\int_C\frac{1}{r_{12}^3}\begin{vmatrix} x_1' & x_2' & x_1-x_2\\ y_1' & y_2' & y_1-y_2\\ z_1' & z_2' & z_1-z_2\end{vmatrix}dt_1dt_2,$$

$$Y=\frac{1}{2\pi}\int_C\frac{ds}{T}; \tag{2}$$

$\vec r_{12}=\vec r_1-\vec r_2, r=|\vec r_{12}|, ds=\sqrt{dx^2+dy^2+dz^2}, 1/T$ is the torsion of the curve ($T=$ the torsion radius) at the current point of integration and primes denote derivatives. Unlike the two terms in the right-hand side (rhs) of Eqs. (2), both of which may take any value, $K$ is always an integer associated with a given curve $C$ of the third order of continuity, and remains constant under deformations of the same class (provided the torsion is finite during deformations). When during the deformation the curve crosses itself, the first (noninteger) term (the double integral) varies with two units, so does $K$ and, therefore, with the aid of $K$ one can judge on the change of the deformation class (in mathematical language, isotopy class). The second term in the rhs of Eqs. (2) [$(1/2\pi)$ integral torsion] is not sensible to self-crossing, but jumps with one unit if during the deformation the torsion $1/T$ is unbounded ($T=0$) in one point.

We shall elaborate now a little bit on a certain critical value of $\varepsilon$. Putting $nv=u$ ($n=q/p$), the torsion of the knot (1) can be expressed in the form

 

$$\frac{1}{T}=\frac{1}{R_T}\frac{(\cos u+1)A+\left(\varepsilon-\frac{1}{n^2+1}\right)B}{(\cos u+1)C+\left(\varepsilon-\frac{1}{n^2+1}\right)^2 D}, \tag{3}$$

where

$$A=-n(n^2-1)\varepsilon^3\cos^2 u+n\varepsilon^2[2(2n^2+1)+(n^2-1)\varepsilon]\cos u$$
$$-n(n^2-1)\varepsilon-2n(2n^2+1)\varepsilon^2+n(2n^4+3n^2+1)\varepsilon^3,$$

$$B=-n(2n^4+3n^2+1)\varepsilon\left(\varepsilon+\frac{n^2-1}{2n^2+1}\right),$$

$$C=(-n^2+1)\varepsilon^4\cos^3 u+\varepsilon^3[4+(n^2-1)\varepsilon]\cos^2 u$$
$$+[3(n^2+2)\varepsilon^2-4\varepsilon^3+(-n^4+3n^2+1)\varepsilon^4]\cos u$$
$$+2(n^2+2)\varepsilon-3(n^2+2)\varepsilon^2+4(n^4+2n^2+1)\varepsilon^3$$
$$+(n^4-3n^2-1)\varepsilon^4,$$

$$D=(n^2+1)^2[(n^2+1)\varepsilon^2-2\varepsilon+1].$$

One sees that if $\cos u$ goes to $-1$ faster than goes $\varepsilon$ to the critical value

$$\varepsilon_c=\frac{1}{(n^2+1)}=\frac{p^2}{p^2+q^2}, \tag{4}$$

then the torsion $1/T$ goes to $\infty$

$$\lim_{\varepsilon\to\varepsilon_c}\left(\lim_{\cos u\to -1}\frac{1}{T}\right)=\infty.$$

There are $q$ critical points $v_k=(2k+1)\pi p/q, k=0,1,2,\ldots,q-1$, on the spiral, where the torsion becomes infinite when $\varepsilon\to\varepsilon_c$, at wich $\cos u=-1$. From Eq. (3) one sees that the behavior of the torsion is very different in the folowing two cases:

(1) In the $q$ points $v_k$ on the spiral, when $\cos u$ is fixed to $-1$ during the deformation (i.e., variation of $\varepsilon\in[0,1]$) the torsion jumps from $\infty$ to $-\infty$,

$$\lim_{\substack{\varepsilon\to\varepsilon_c\\(\varepsilon<\varepsilon_c)}}\left(\frac{1}{T}\right)(\cos u=-1,\varepsilon)=\infty,$$

$$\lim_{\substack{\varepsilon\to\varepsilon_c\\(\varepsilon>\varepsilon_c)}}\left(\frac{1}{T}\right)(\cos u=-1,\varepsilon)=-\infty.$$

This infinite jump is responsible for the variation of the invariant $K$ by $q$ units during the deformation.

(2) If from all spirals corresponding to various $\varepsilon$, one singles out the one corresponding to $\varepsilon_c$ given by Eq. (4) (let us call it the critical spiral), the situation is different with respect to case (1) for $\varepsilon\neq\varepsilon_c$: once $\varepsilon$ is fixed at $\varepsilon_c$, Eq. (3) tells us that

$$\lim_{\cos u\to -1}\left(\frac{1}{T}\right)(u,\varepsilon=\varepsilon_c)=finite.$$

So the critical spiral is a gentle curve, without discontinuities of the torsion. Calugareanu's self-linking invariant (integer number) has, therefore, this remarkable property: it isolates from the family of curves in the deformation process the well-behaved critical curve, devoid of torsion discontinuities and with a possible half-integer value for $K$. The critical spiral being a well-behaved curve, it makes sense to calculate $K$ for it also, although the theory cannot tell whether in this case $K$ continues to be an integer. We have found numerical support for a formula of the type

$$K(\varepsilon=\varepsilon_c)=\tfrac{1}{2}[K(\varepsilon<\varepsilon_c)+K(\varepsilon>\varepsilon_c)],$$

but with $K(\varepsilon=\varepsilon_c)$ not always an integer.

The integral torsion of the spiral Eqs. (1) can be computed analytically and so one finds

$$Y=\frac{1}{2\pi}\int_C\frac{ds}{T}=\begin{cases}\frac{\sigma}{2\pi}+q & \text{for}\quad \varepsilon<\varepsilon_c,\\ \frac{\sigma}{2\pi} & \text{for}\quad \varepsilon>\varepsilon_c,\end{cases} \tag{5}$$

$$\sigma=-4q\{[n^2\varepsilon^2+(1+\varepsilon)^2][n^2\varepsilon^2+(1-\varepsilon)^2]\}^{-1/4}K(k), \tag{6}$$

with $K(k)$ being the complete elliptic integral of argument $k$ given by

$$k^2=\frac{1}{2}-\frac{1}{2}\frac{1-\varepsilon^2+n^2\varepsilon^2}{\{[n^2\varepsilon^2+(1+\varepsilon)^2][n^2\varepsilon^2+(1-\varepsilon)^2]\}^{1/2}}. \tag{7}$$

One remarks so the jump of $q$ units of the second term $Y$ in the rhs of Calugareanu's formula (2) at the critical point $\varepsilon_c$. We have got then an exact formula for the first term $J$, the double integral over the spiral given by Eqs. (1),

$$J=-qp+\frac{2q}{\pi}\{[n^2\varepsilon^2+(1+\varepsilon)^2]$$
$$\times[n^2\varepsilon^2+(1-\varepsilon)^2]\}^{(-1/4)}K(k). \tag{8}$$

One sees that $J$ is not sensible to the critical value $\varepsilon_c$, exactly as the Calugareanu theory says. When the concrete form of Eqs. (1) for the toroidal spiral is inserted in the expression of $J$ from its definition Eqs. (2), Eq. (8) provides a nice identity that we have verified numerically to a good accuracy. The Calugareanu invariant $K$ for the toroidal spiral Eqs. (1) is, therefore,

$$K=\begin{cases}-qp+q & \text{for}\quad \varepsilon<\varepsilon_c,\\ -qp & \text{for}\quad \varepsilon>\varepsilon_c,\end{cases} \tag{9}$$

again in agreement with the theory [1] that implies a jump of $q$ units for $K$ when $\varepsilon_c$ is crossed. There is only one torus, namely, the one with $\varepsilon = r_T/R_T = \varepsilon_c = p^2/(p^2+q^2)$, whose $(p,q)$ spiral, a ''well-behaved'' curve (that is devoid of torsion discontinuities, i.e., $T \neq 0$), marks the change of the isotopy class without self-crossing, but on account of torsion jumps. In this sense, there is one torus more torus than the other tori: the critical one, the one with $(r_T/R_T) = p^2/(p^2+q^2)$.

We have computed also other Calugareanu invariants (of second order of regularity, etc.) for the toroidal spiral. We expect *mathematical* objects that happen to jump by integers in different *physical* situations (e.g., when the torus is thin or thick) to lead somewhere. The contrary would be curious but important as well.

Now, let us put a (generally time dependent) current of intensity $I(t)$ into the toroidal spiral. Our task is to calculate the toroid and magnetic multipole moments (and the radii of any order of their corresponding distributions). All these quantities realize a full description of our specific current distribution. We shall present here only the minimum of details. We have evaluated the toroid radii of order $s$ and multipolarity $l$,

$$R_{lm}^{2s}(t) = -\frac{1}{c(2l+1)}\sqrt{\frac{4\pi l}{l+1}}\int d^3r r^{l+2s+1} \times\left[\frac{\sqrt{l}}{(2l+2s+3)}\vec{Y}^{*}_{l,l+1,m}(\theta,\varphi) + \frac{\sqrt{l+1}}{2(s+1)}\vec{Y}^{*}_{l,l-1,m}(\theta,\varphi)\right]\vec{j}(\vec{r},t), \tag{10}$$

as well as the magnetic radii

$$\rho_{lm}^{2s}(t) = \frac{1}{ic}\sqrt{\frac{4\pi l}{(l+1)(2l+1)}}\int d^3r r^{2s+l}\vec{Y}^{*}_{llm}(\theta,\varphi)\vec{j}(\vec{r},t) \tag{11}$$

($\vec{Y}$ are the usual vector spherical harmonics), by using for the current density $\vec{j}(\vec{r},t)$ the line integral

$$\vec{j}(\vec{r},t) = I(t)\int_C ds'\frac{d\vec{r}'(s')}{ds'}\frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{r^2\sin\theta}, \tag{12}$$

corresponding to the linear current flowing through the toroidal spiral (1), specified above by the contour $C$. For zero order $(s=0)$, from Eqs. (10), (11) one gets the multipole toroid and magnetic moments of the knotted current. As shown in Ref. [3], $\rho_{lm}^{2s}(t)$ $(s=0,1,2,\ldots)$ give rise to magnetic type radiation (the usual $Ml$ waves) while the toroid moments and radii $R_{lm}^{2s}(t)$, to the usual electric type $El$ waves. Toroid sources emit electric type radiation. Sources, unlike radiation, are described by three families of multipoles (the electric, magnetic, and toroid ones), while their radiation is only of two types, the usual electric ($El$) and magnetic ($Ml$) waves. In our case, although there are no electric multipole moments, $El$ waves are emitted on account of the toroid multipoles. The ''unusual'' toroid dipole (anapole) studied in Refs. [2–6] is $R_{lm}^{2s}(t)$ for $s=0$, $l=1$, $m=-1,0,1$, i.e., the vector (in spherical basis) $R^0_{1,m=-1,0,1}(t)$, in most Cartesian notations $\vec{T}(t) = (1/10c)\int d^3r\{\vec{r}[\vec{r}\cdot\vec{j}(\vec{r},t)] - 2\vec{r}^2\vec{j}(\vec{r},t)\}$.

With Eqs. (10), (11), (12), going from vector spherical harmonics to the Legendre associated functions $P_l^m$, one finds

$$i(-1)^m\frac{c}{I(t)}\frac{2(l+1)}{R_T^{2s+l+1}}\sqrt{\frac{(l+m)!}{(l-m)!}}\rho_{lm}^{2s}(t) = \int_0^{2p\pi}dv[1+\varepsilon^2+2\varepsilon\cos(nv)]^{s+(l/2)} \tag{13}$$

$$[(l+m)(l-m+1)P_l^{m-1}(\cos\theta)U_m + P_l^{m+1}(\cos\theta)U^{*}_{-m} + mnP_l^m(\cos\theta)V_m],$$

where

$$\cos\theta = \frac{\varepsilon\sin(nv)}{\sqrt{1+\varepsilon^2+2\varepsilon\cos(nv)}},$$

$$U_m = ie^{-imv}\left[1-(n-1)\frac{\varepsilon}{2}e^{inv}+(n+1)\frac{\varepsilon}{2}e^{-inv}\right],$$

$$V_m = 2\varepsilon e^{-imv}\cos(nv),\rho_{l,-m}^{2s} = (-1)^m\rho_{l,m}^{2s*},$$

and a similar expression for the toroid radii of any multipolarity and order $R_{lm}^{2s}(t)$. The formula (13) and its toroid analog are exact and perhaps on the basis of such results something serious about the topological implications discussed in the first part of this paper (critical dependences on $\varepsilon_c$) should come out. Unfortunately, from now on we shall work in the approximation of the thin torus (i.e., to order $\varepsilon^2$). We first calculate the radii, getting results of the type

$$\rho_{l,m=0}^{2s}(t) = \frac{\pi^{3/2}I(t)R_T^{2s+l+1}p\left[2+(4s^2+4sl+4s+l+1)\frac{\varepsilon^2}{2}\right]}{c\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+3}{2}\right)}, \tag{14}$$

$$R_{l,m=0}^{2s}(t) = \frac{2\pi^{3/2}I(t)R_T^{2s+l+2}q\varepsilon^2}{c(l+1)\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}, \tag{15}$$

and somewhat more complicated for $m \geqslant 1$, then succeed to sum up the series

$$M_{lm}(-k^2,t) = \sum_{s=0}^{\infty}\frac{(-k^2)^s}{s!}M_{lm}^{(s)}(0,t),$$

$$\rho_{lm}^{2s}(t)=\frac{2^s(2l+2s+1)!!}{(2l+1)!!}M_{lm}^{(s)}(0,t) \tag{16}$$

(and its toroid analog exactly of the same type) to get the magnetic and toroid formfactors for the linear current flowing through the toroidal knot in the approximation of the thin torus (to the second order in $\varepsilon=r_T/R_T$)

$$\begin{aligned}M_{l,m=0}(-k^2,t)&=\frac{p\pi^2 I(t)R_T^{l+1}(2l+1)!!}{c\sqrt{2}\Gamma\left(\frac{-l}{2}\right)\Gamma\left(\frac{l+3}{2}\right)}\\&\times\Big\{2(kR_T)^{-l-(1/2)}J_{l+(1/2)}(kR_T)\\&+\frac{\varepsilon^2}{2}[(-l^2-l+\tfrac{1}{4})(kR_T)^{-l-(1/2)}J_{l+(1/2)}(kR_T)\\&+2(kR_T)^{-l+(1/2)}J'_{l+(1/2)}(kR_T)\\&+(kR_T)^{-l+(3/2)}J''_{l+(1/2)}(kR_T)]\Big\},\end{aligned} \tag{17}$$

$$T_{l,m=0}(-k^2,t)=\frac{q\pi^2\varepsilon^2\sqrt{2}I(t)R_T^{l+2}(2l+1)!!}{c(l+1)\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}\times(kR_T)^{-l-(1/2)}J_{l+(1/2)}(kR_T), \tag{18}$$

and similar results for $m\geqslant 1$. $J_l$ = cylindrical Bessel functions, $M_{l,-m}=(-1)^m M_{l,m}^*$ and analogously for $T_{l,-m}$.

Equations (17), (18) express all information about the electromagnetic structure of the knotted toroidal linear current (in the approximation of the thin torus, to order $\varepsilon^2$). The appearance of $p$ and $q$ factors in these formulas is significant, albeit expected. From them, one can simply get the behavior of the magnetic and toroid formfactors at small and high frequencies. $M_{lm}$ and $T_{lm}$ all go to constants at low frequencies, e.g.,

$$M_{l,m=0}(k^2,t)\underset{k\to 0}{\sim}\frac{p\pi^{3/2}I(t)R_T^{l+1}}{c\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+3}{2}\right)}\left[2+\frac{\varepsilon^2}{2}(l+1)\right]=M_{l,m=0}(0,t), \tag{19}$$

$$T_{l,m=0}(k^2,t)\underset{k\to 0}{\sim}\frac{q2\varepsilon^2\pi^{3/2}I(t)R_T^{l+2}}{c(l+1)\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}=T_{l,m=0}(0,t), \tag{20}$$

while at high frequencies one gets formulas of the type

$$M_{l,m=0}(k^2,t)\underset{k\to\infty}{\sim}\frac{p\pi^{3/2}r_T^2 I(t)}{2c}\frac{(2l+1)!!}{\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+3}{2}\right)}\times\frac{\cos\left[kR_T-(l+1)\frac{\pi}{2}\right]}{k^{l-1}}, \tag{21}$$

$$T_{l,m=0}(k^2,t)\underset{k\to\infty}{\sim}\frac{q2\varepsilon^2\pi^{3/2}R_T I(t)}{c(l+1)}\frac{(2l+1)!!}{\Gamma\left(-\frac{l}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}\times\frac{\sin\left(kR_T-\frac{l\pi}{2}\right)}{k^{l+1}}. \tag{22}$$

For $m\geqslant 1$, analogous but more complicated formulas were obtained. When the current $I(t)$ is harmonic $I(t)=I\cos\omega t$, we have computed the radiation of such a toroidal linear current, this time not restricting the analysis to thin tori, but working for any $\varepsilon\in[0,1]$ in the approximation of the toroidal spiral seen from far away and for wavelengths larger than the system's dimensions. We calculated the fields to order $k^3$, $k=\omega/c$, and got for the radiation intensity $\mathcal{I}$ (for large $n=q/p$) at low frequencies

$$\mathcal{I}=\frac{p^2\pi^2 I^2 k^4}{3c}\left(R_T^2+\frac{r_T^2}{2}\right)^2. \tag{23}$$

It is a magnetic type radiation; the electric type radiation (marked by factors containing $q$) coming from toroid sources starts appearing in the next orders in frequency.

During the past decade there has been research done on radiation and scattering by knotted structures. For a sampling of such papers, see Refs. [7] and [8].

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